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SYMBOLIC COMPUTATION OF FLOW IN A ROTATING PIPE

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SUMMARY

A perturbation solution of the fully developed flow through a pipe of circular cross-section, which rotates uniformly around an axis oriented perpendicularly to its own, is considered. The perturbation parameter is given by $R = 2\Omega a^2/\nu$ in terms of the angular velocity Ω, the pipe radius a and the kinematic viscosity ν of the fluid. The two coupled non-linear equations for the axial velocity ω and the streamfunction ϕ of the transverse (secondary) flow lead to an infinite system of linear equations. This system allows first the computation of a given order ϕ_n, $n \geq 1$, of the perturbation expansion $\phi = \sum_{n=1}^{\infty} R^n \phi_n$ in terms of ω_{n-1}, the $(n-1)$-th order of the expansion $\omega = \sum_{n=0}^{\infty} R^n \omega_n$, and of the lower orders $\phi_1, \ldots, \phi_{n-1}$. Then it permits the computation of ω_n from $\omega_0, \ldots, \omega_{n-1}$ and $\phi_1, \ldots, \phi_n$. The computation starts from the Hagen–Poiseuille flow ω_0, i.e. the perturbation is around this flow.

The computations are performed analytically by computer, with the REDUCE and MAPLE systems. The essential elements for this are the appropriate co-ordinates: in the complex co-ordinates chosen the two-dimensional harmonic (Laplace, Δ) and biharmonic (Δ^2) operators are ideally suited for (symbolic) quadratures. Symmetry considerations as well as analysis of the equations for ω_n, ϕ_n and of the boundary conditions lead to general (polynomial) formulae for these functions, with coeffcients to be determined. Their determination, order by order, implies, in complex co-ordinates, only (symbolic) differentiation and quadratures. The coefficients themselves are polynomials in the Reynolds number c of the (unperturbed) Hagen–Poiseuille flow. They are tabulated in the paper for the orders $n \leq 6$ of the perturbation expansion.

KEY WORDS Computer algebra Pipe flow Rotating pipe Perturbation expansion

1. INTRODUCTION

The flow of a viscous fluid through a straight pipe which rotates about an axis perpendicular to its own is of importance in many fields of engineering. It gives on the practical side, for example, important background to the measurement of mass flow rate by the Coriolis effect.[1] The instruments which exploit the inertial (Coriolis) reaction of the flowing fluid on the forcedly vibrating pipe are now required to work with an accuracy of a fraction of a per cent. By this, one is forced in the analysis of the instrument already to the level of the finer flow effects, those produced

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by the secondary flow of the fluid on the vibration of the pipe. On the other hand the flow is also of considerable interest from the point of view of basic fluid mechanics, in particular related to the nature of the secondary flow and to its stability.

At present computational results on this flow are available mainly numerically;[2,3] analytic results[4,5] are, at present standards, very modest, although they are expected to offer important insight in addition to numerical work. The reason for this lies in the fact that their derivation by hand gets very tedious and laborious; it becomes also very hard to avoid computational mistakes.

It is the aim of the present paper to approach the flow in a rotating pipe anew in an analytic way, from a perspective which takes into account the advanced symbolic computation facilities (e.g. REDUCE, MAPLE or MACSYMA) available now. These facilities allow one to develop systematically the idea of perturbation[4] of the Hagen–Poiseuille flow due to rotation and to devise an analytic scheme for the computation of the flow by successive approximations. The scheme is described in the paper.

The structure of the paper is as follows. In Section 2 the equations of the fluid flow in a rotating reference frame, which is the natural one to be chosen for this flow, are reviewed and formulated, in terms of a streamfunction, for a fully developed flow (i.e. translationally invariant along the pipe axis) in a circular pipe, when the pipe rotates uniformly around an axis perpendicular to its own. Then (Section 3) the coupled nonlinear equations for the streamfunction ϕ and the axial velocity ω are transformed for convenience to complex co-ordinates.* The perturbation procedure around the Hagen–Poiseuille solution for the non-rotating pipe, with a dimensionless rotation number R as perturbation parameter, is then considered for the solution of this system. The two non-linear equations thereby lead to a (doubly) infinite system of linear equations for the various orders ω_n, ϕ_n ($n = 0, 1, 2, \ldots$) in the perturbation expansion of ω, ϕ. The structure of the equations is governed by the Laplace operator Δ for ω and by its square Δ^2, the biharmonic operator, for ϕ. These operators take in complex co-ordinates particularly simple forms which are most suitable for symbolic integration of the equations.

The solutions have symmetry properties. One obvious symmetry is that with respect to reflection in the central pipe plane perpendicular to the axis of rotation. Although there is no reflection symmetry of the flow with respect to the plane of the pipe and rotation axes, the various orders ω_n, ϕ_n of perturbation do have such symmetries, which are alternating with n: ω_{2n} and ϕ_{2n+1} are symmetric whereas ω_{2n+1} and ϕ_{2n} are antisymmetric. However, if one changes the sign of rotation ($R \to -R$) simultaneously with this reflection, then ω behaves symmetrically and ϕ antisymmetrically under this combined operation.

In Section 4 the method of solution of the system of equations for ω_n, ϕ_n ($n = 0, 1, 2, \ldots$) is described in such a way that it can be performed symbolically by a computer, e.g. with REDUCE[6,7] or MAPLE.[8] Thereby ϕ_n, for a given order $n \geq 1$, is computed essentially by quadratures from ω_{n-1} (ω_0 is the Hagen–Poiseuille sollution) and $\phi_1, \ldots, \phi_{n-1}$ ($\phi_0 = 0$), and afterwards ω_n is computed similarly from $\omega_0, \ldots, \omega_{n-1}$ and $\phi_1, \ldots, \phi_n$. Since the start function ω_0 is a (second-order) polynomial in the complex variables, and because differentiation and quadratures keep the polynomial structure, the functions ω_n, ϕ_n are polynomials in these variables. Several of their properties can be derived directly from the equations and are exploited then during the computations. In fact, the information on the polynomials ω_n, ϕ_n from their equations leads, together with the boundary conditions, to certain structure formulae for them, wherein the coefficients have to be determined then, analytically, by the equations. The results of the computation (Section 5) are conveniently given by tables of these polynomial coefficients.

* The notation ϕ of Barua[4] is followed here for the streamfunction instead of the conventional notation ψ.

They are again polynomials, in the Reynolds number of the Hagen–Poiseuille flow. For the purpose of illustration, the coefficients of $\omega_0, \ldots, \omega_6$ and $\phi_1, \ldots, \phi_6$ are presented in Tables I and II respectively.

2. FLUID EQUATIONS IN A ROTATING FRAME

In the present section the equations of the fluid flow in the rotating reference frame will be briefly reviewed. The inertial reference frame will be characterized by the orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the rotating frame by the angular velocity $\mathbf{\Omega} = \Omega\mathbf{e}_1$ oriented along the $\mathbf{e}_1$-axis. The co-ordinate axes of this frame will be denoted by $\mathbf{f}_1(t)$, $\mathbf{f}_2(t)$, $\mathbf{f}_3(t)$ with $\mathbf{f}_1(t) = \mathbf{e}_1$. To simplify the problem, a stationary flow in the rotating frame will be considered, i.e. $\partial V_i/\partial t = 0$, where

$$\mathbf{V} = V_i\mathbf{f}_i \tag{1}$$

is the velocity field; the co-ordinate vector will be $\mathbf{X} = X_i\mathbf{f}_i$ (summation over repeated indices is assumed). The pipe is considered to extend along the $\mathbf{f}_3$-axis and to have a radius a.

The dynamical equations describing the flow are[9]

$$(\mathbf{V}\cdot\nabla)\mathbf{V} - 2\mathbf{V}\times\mathbf{\Omega} = -\nabla\chi + \nu\Delta\mathbf{V}, \tag{2}$$

where the notation

$$\chi(\mathbf{X}) = \frac{P(\mathbf{X})}{\rho} - \tfrac{1}{2}(\mathbf{\Omega}\times\mathbf{X})^2 \tag{3}$$

has been used; the centrifugal force $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{X})$ has been included in the pressure term by the corresponding potential. Herein $P(\mathbf{X})$ is the pressure and ρ and ν are the fluid density and kinematic viscosity respectively.

An additional assumption on the flow will be made, namely that it is translationally invariant, i.e. the velocity field is independent of the axial co-ordinate X_3; therefore the pressure term will be of the form

$$\chi(\mathbf{X}) = -4c\frac{\nu^2}{a^3}X_3 + \frac{\nu^2}{a^2}\psi(X_1, X_2), \tag{4}$$

with c a dimensionless constant and ψ a dimensionless function. The significance of c is given by $c = 2w_m a/\nu$, where w_m is the mean Hagen–Poiseuille velocity; c is thus the mean Reynolds number of the unperturbed flow.* The equations of motion for the Cartesian components V_i are then written with respect to the co-ordinates X_i as

$$\left(V_1\frac{\partial}{\partial X_1} + V_2\frac{\partial}{\partial X_2}\right)V_1 = -\frac{\partial\chi}{\partial X_1} + \nu\Delta V_1, \tag{5}$$

$$\left(V_1\frac{\partial}{\partial X_1} + V_2\frac{\partial}{\partial X_2}\right)V_2 - 2\Omega V_3 = -\frac{\partial\chi}{\partial X_2} + \nu\Delta V_2, \tag{6}$$

$$\left(V_1\frac{\partial}{\partial X_1} + V_2\frac{\partial}{\partial X_2}\right)V_3 + 2\Omega V_2 = +4c\frac{\nu^2}{a^3} + \nu\Delta V_3, \tag{7}$$

* The unconventional notation c for the Reynolds number Re follows Barua;[4] his parameter c is equal to $-4c$ of the present paper.

with the boundary conditions requiring that the fluid velocity vanishes on the pipe surface in the rotating frame.

One has to add, for an incompressible fluid (ρ = constant), the continuity equation

$$\frac{\partial V_1}{\partial X_1} + \frac{\partial V_2}{\partial X_2} = 0. \tag{8}$$

In the following, dimensionless co-ordinates $x_i = X_i/a$ will be used and dimensionless functions, the streamfunction $\phi(x_1, x_2)$ of the transverse flow and the velocity component along the $\mathbf{f}_3$-axis, $\omega(x_1, x_2)$, will be introduced by

$$V_1 = -\frac{\nu}{a}\frac{\partial \phi}{\partial x_2}, \tag{9}$$

$$V_2 = \frac{\nu}{a}\frac{\partial \phi}{\partial x_1}, \tag{10}$$

$$V_3 = \frac{\nu}{a}\omega; \tag{11}$$

by this definition the continuity equation will be automatically satisfied.

From (5)–(7) the following system of equations for the streamfunction $\phi(x_1, x_2)$ and velocity $\omega(x_1, x_2)$ with respect to the dimensionless Cartesian co-ordinates x_1, x_2 is obtained:[4]

$$\Delta^2 \phi = \left(\frac{\partial \phi}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial \phi}{\partial x_2}\frac{\partial}{\partial x_1}\right)\Delta\phi - R\frac{\partial \omega}{\partial x_1}, \tag{12}$$

$$\Delta \omega = \left(\frac{\partial \phi}{\partial x_1}\frac{\partial \omega}{\partial x_2} - \frac{\partial \phi}{\partial x_2}\frac{\partial \omega}{\partial x_1}\right) + R\frac{\partial \phi}{\partial x_1} - 4c, \tag{13}$$

where $R = 2\Omega a^2/\nu$ is a dimensionless constant (a Taylor number) and $\Delta^2 = \Delta\Delta$, with

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{14}$$

is the transverse Laplace operator. The properties of ϕ and ω will be discussed in the following sections. The pressure $\chi(x_1, x_2)$ will not be considered in this paper; it can be obtained directly by quadrature from equations (5) and (6).

3. THE PROBLEM IN TERMS OF COMPLEX CO-ORDINATES

The computation of the flow can be approached at present generally only by approximation procedures of either an analytic or a numerical nature. The geometry of the problem suggests the use of cylindrical co-ordinates. Formulating the problem in these co-ordinates, Barua has sought a solution by perturbation of the Hagen–Poiseuille flow;[4,5] since the expansion parameter is $R = 2\Omega a^2/\nu$, the orders of the perturbation expansion needed for the description of the flow will increase with the increase of the angular velocity. The two lowest orders of the expansion have been calculated almost completely, analytically, by Barua; higher-order calculations have been performed, in certain simplifying limit situations, only by numerical methods.[2,3] The difficulties of the analytic calculation in the cylindrical co-ordinates are, however, considerable. Therefore an alternative approach is introduced in the present paper in order to simplify the form of equations (12) and (13). This then allows the extension of the analytic calculation of the perturbation solution to arbitrary higher orders by means of computer algebra (symbolic computation). The procedure is based on the observation that the use of the dimensionless complex variables

$$z = x_1 + \mathrm{i}x_2, \tag{15}$$

$$z_* = x_1 - \mathrm{i}x_2 \tag{16}$$

leads to a tremendous simplification of the form of the Laplace operator.[10] One has, obviously, $z = re^{i\theta}$ and $z_* = re^{-i\theta}$, where r, θ, x_3 are dimensionless cylindrical co-ordinates.

In the variables z, z_* the system of equations (12), (13) is

$$\Delta^2\phi(z, z_*) = -2\mathrm{i}L(\phi(z, z_*), \Delta\phi(z, z_*)) - R\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z_*}\right)\omega(z, z_*), \tag{17}$$

$$\Delta\omega(z, z_*) = -2\mathrm{i}L(\phi(z, z_*), \omega(z, z_*)) + R\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z_*}\right)\phi(z, z_*) - 4c, \tag{18}$$

where the notation

$$L(F, G) = \frac{\partial F}{\partial z}\frac{\partial G}{\partial z_*} - \frac{\partial G}{\partial z}\frac{\partial F}{\partial z_*} \tag{19}$$

has been used; the mentioned simple form of the Laplace operator is

$$\Delta = 2^2\frac{\partial^2}{\partial z \partial z_*}, \tag{20}$$

and its square, accordingly,

$$\Delta^2 = 2^4\frac{\partial^4}{\partial z^2 \partial z_*^2}. \tag{21}$$

On the pipe surface ($zz_* = 1$) the boundary conditions

$$\frac{\partial\phi}{\partial z} = 0 \qquad (zz_* = 1), \tag{22}$$

$$\frac{\partial\phi}{\partial z_*} = 0 \qquad (zz_* = 1), \tag{23}$$

$$\omega = 0 \qquad (zz_* = 1) \tag{24}$$

have to be satisfied. The conditions (22) and (23) are, up to an irrelevant constant, equivalent to

$$\phi = 0 \qquad (zz_* = 1), \tag{25}$$

$$\frac{\partial\phi}{\partial r} = 0 \qquad (zz_* = 1). \tag{26}$$

Equations (17) and (18) show the following symmetry properties of the solutions $\phi(z, z_*)$, $\omega(z, z_*)$ with respect to the reflection $x_1 \to -x_1, x_2 \to x_2$, i.e. $z \to -z_*, z_* \to -z$:

$$\phi(-z_*, -z) = -\phi(z, z_*), \tag{27}$$

$$\omega(-z_*, -z) = \omega(z, z_*). \tag{28}$$

There is also a symmetry with respect to the reflection $x_1 \to x_1, x_2 \to -x_2$, i.e. $z \to z_*, z_* \to z$, and a simultaneous change of the sense of rotation ($\Omega \to -\Omega$):

$$\phi(z_*, z; -R) = -\phi(z, z_*; R), \tag{29}$$

$$\omega(z_*, z; -R) = \omega(z, z_*; R). \tag{30}$$

These relations connect in fact the flow fields in two oppositely rotating pipes. Therefore the dependence of ϕ and ω on the Taylor numbers $\pm R$ has been explicitly indicated here.

Finally, there is as an additional general property of $\phi(z, z_*)$ and $\omega(z, z_*)$, their reality:

$$\phi^*(z, z_*) = \phi(z, z_*), \tag{31}$$

$$\omega^*(z, z_*) = \omega(z, z_*). \tag{32}$$

Following Barua,[4] the solutions ϕ, ω of equations (17), (18) will be sought as series expansions

$$\phi(z, z_*) = \sum_{n=0}^{\infty} R^n \phi_n(z, z_*), \tag{33}$$

$$\omega(z, z_*) = \sum_{n=0}^{\infty} R^n \omega_n(z, z_*), \tag{34}$$

wherein the lowest order is given by the Hagen–Poiseuille flow in a non-rotating pipe:

$$\phi_0 = 0, \tag{35}$$

$$\omega_0 = c(1 - zz_*). \tag{36}$$

This perturbation procedure produces from the two non-linear equations (17), (18) an infinite system of linear equations for the functions $\phi_n(z, z_*)$ and $\omega_n(z, z_*)$, $n \geq 1$:

$$\Delta^2 \phi_n(z, z_*) = -2\mathrm{i} \sum_{k=1}^{n-1} L(\phi_k(z, z_*), \Delta\phi_{n-k}(z, z_*)) - \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z_*}\right)\omega_{n-1}(z, z_*), \tag{37}$$

$$\Delta\omega_n(z, z_*) = -2\mathrm{i} \sum_{k=1}^{n} L(\phi_k(z, z_*), \omega_{n-k}(z, z_*)) + \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z_*}\right)\phi_{n-1}(z, z_*), \tag{38}$$

with the boundary conditions

$$\frac{\partial \phi_n}{\partial z} = 0 \qquad (zz_* = 1), \tag{39}$$

$$\frac{\partial \phi_n}{\partial z_*} = 0 \qquad (zz_* = 1), \tag{40}$$

or equivalently

$$\phi_n = 0 \qquad (zz_* = 1), \tag{41}$$

$$\frac{\partial \phi_n}{\partial r} = 0 \qquad (zz_* = 1), \tag{42}$$

and

$$\omega_n = 0 \qquad (zz_* = 1). \tag{43}$$

The functions $\phi_n(z, z_*)$, $\omega_n(z, z_*)$ have symmetry properties which follow from (27)–(30):

$$\phi_n(-z_*, -z) = -\phi_n(z, z_*), \tag{44}$$

$$\omega_n(-z_*, -z) = \omega_n(z, z_*) \tag{45}$$

and

$$\phi_n(z_*, z) = (-1)^{n-1}\phi_n(z, z_*), \tag{46}$$

$$\omega_n(z_*, z) = (-1)^n\omega_n(z, z_*) \tag{47}$$

respectively. From reality of ϕ, ω also their reality follows:

$$\phi_n^*(z, z_*) = \phi_n(z, z_*), \tag{48}$$

$$\omega_n^*(z, z_*) = \omega_n(z, z_*). \tag{49}$$

The properties (44)–(49) restrict the form of the functions $\phi_n(z, z_*)$, $\omega_n(z, z_*)$ and therefore can be used either as part of the input to the calculations or as an additional test for their correctness in each of the successive orders of approximation n.

4. SOLUTION OF THE PROBLEM

For each $n \geq 1$ the solutions of the system of (non-homogeneous) equations (37), (38) can be split as

$$\phi_n(z, z_*) = \phi_n^{(1)}(z, z_*) + \phi_n^{(0)}(z, z_*), \tag{50}$$

$$\omega_n(z, z_*) = \omega_n^{(1)}(z, z_*) + \omega_n^{(0)}(z, z_*), \tag{51}$$

where $\phi_n^{(1)}(z, z_*)$, $\omega_n^{(1)}(z, z_*)$ are particular solutions of the non-homogeneous equations and $\phi_n^{(0)}(z, z_*)$, $\omega_n^{(0)}(z, z_*)$ are then solutions of the corresponding homogeneous equations. Because of the simple form of the operator Δ (or Δ^2 respectively) in the co-ordinates z, z_*, particular solutions $\phi_n^{(1)}(z, z_*)$, $\omega_n^{(1)}(z, z_*)$ can be found directly by quadratures from (37) and (38) in terms of the functions $\phi_k(z, z_*)$, $\omega_k(z, z_*)$ calculated previously (at the steps $k < n$).

Then $\omega_n^{(0)}(z, z_*)$ is harmonic, the solution of the Dirichlet problem

$$\Delta\omega_n^{(0)}(z, z_*) = 0, \tag{52}$$

with the boundary condition

$$\omega_n^{(0)}(z, z_*) = -\omega_n^{(1)}(z, z_*) \quad (zz_* = 1). \tag{53}$$

It must have, as a real function, the following convergent series expansion:

$$\omega_n^{(0)}(z, z_*) = a_0 + a_1 z + a_2 z^2 + \ldots + a_0^* + a_1^* z_* + a_2^* z_*^2 + \ldots \tag{54}$$

where $a_0, a_1, a_2, \ldots$ are coefficients to be determined. Their determination is possible due to the uniqueness of the solution of the Dirichlet problem.[10]

On the other hand $\phi_n^{(0)}(z, z_*)$ is a biharmonic function, the solution of the biharmonic equation

$$\Delta^2\phi_n^{(0)}(z, z_*) = 0, \tag{55}$$

with the boundary conditions

$$\frac{\partial\phi_n^{(0)}(z, z_*)}{\partial z} = -\frac{\partial\phi_n^{(1)}(z, z_*)}{\partial z} \qquad (zz_* = 1), \tag{56}$$

$$\frac{\partial\phi_n^{(0)}(z, z_*)}{\partial z_*} = -\frac{\partial\phi_n^{(1)}(z, z_*)}{\partial z_*} \qquad (zz_* = 1). \tag{57}$$

By the equivalence of (39), (40) and (41), (42), $\phi_n^{(0)}(z, z_*)$ is actually the solution of a Dirichlet–Neumann problem[10,11] for the biharmonic equation (55) and can thereby be re-presented as a superposition of the following kind, of two harmonic functions $u_1(z, z_*)$, $u_2(z, z_*)$, each of the form (54):[10]

$$\phi_n^{(0)}(z, z_*) = zz_* u_1^{(0)}(z, z_*) + u_2^{(0)}(z, z_*). \tag{58}$$

The boundary conditions (56), (57) determine uniquely the coefficients for both functions $u_1^{(0)}(z, z_*)$, $u_2^{(0)}(z, z_*)$ and consequently the function $\phi_n^{(0)}(z, z_*)$.[10,11] From the structure of equations (37), (38) and from the properties (44)–(49) one can derive more precise statements on the form of the functions $\omega_n^{(1)}(z, z_*)$ and $\phi_n^{(1)}(z, z_*)$ obtained by quadratures. The function $\omega_n^{(1)}(z, z_*)$ is a polynomial of z, z_*, of degree $5n + 2$, with the properties

$$\omega_n^{(1)}(z, z_*) = \omega_n^{(1)}(-z_*, -z), \tag{59}$$

$$\omega_n^{(1)}(z, z_*) = (-1)^n \omega_n^{(1)}(z_*, z), \tag{60}$$

$$\omega_n^{(1)}(z, z_*) = \omega_n^{(1)*}(z, z_*), \tag{61}$$

where $\phi_n^{(1)}(z, z_*)$ is a polynomial of z, z_*, of degree $5n$, with the properties

$$\phi_n^{(1)}(z, z_*) = -\phi_n^{(1)}(-z_*, -z), \tag{62}$$

$$\phi_n^{(1)}(z, z_*) = -(-1)^n \phi_n^{(1)}(z_*, z), \tag{63}$$

$$\phi_n^{(1)}(z, z_*) = \phi_n^{(1)*}(z, z_*). \tag{64}$$

This implies that $\omega_{2m}^{(1)}(z, z_*)$ contains only factors of the form $z^{2l} + z_*^{2l}$ and $(zz_*)^k$, whereas $\omega_{2m+1}^{(1)}(z, z_*)$ contains only $i(z^{2l+1} - z_*^{2l+1})$ and $(zz_*)^k$. The highest possible value of l thereby turns out to be $l = m$.

The form of $\omega_n^{(1)}(z, z_*)$ is thus

$$\omega_{2m}^{(1)}(z, z_*) = \sum_{l=0}^{m} (z^{2l} + z_*^{2l}) \sum_{k=0}^{5m-l+1} a_k^{(l)} (zz_*)^k, \tag{65}$$

$$\omega_{2m+1}^{(1)}(z, z_*) = i \sum_{l=0}^{m} (z^{2l+1} - z_*^{2l+1}) \sum_{k=0}^{5m-l+3} a_k^{(l)} (zz_*)^k, \tag{66}$$

with real coefficients $a_k^{(l)}$.

Since $\omega_n^{(0)}(z, z_*)$ has the symmetry properties

$$\omega_n^{(0)}(z, z_*) = \omega_n^{(0)}(-z_*, -z), \tag{67}$$

$$\omega_n^{(0)}(z, z_*) = (-1)^n \omega_n^{(0)}(z_*, z), \tag{68}$$

$$\omega_n^{(0)}(z, z_*) = \omega_n^{(0)*}(z, z_*), \tag{69}$$

and is in addition harmonic, it has to be of the form (65), (66), but only with the terms $k = 0$, i.e.

$$\omega_{2m}^{(0)}(z, z_*) = \sum_l b^{(l)} (z^{2l} + z_*^{2l}), \tag{70}$$

$$\omega_{2m+1}^{(0)}(z, z_*) = i \sum_l b^{(l)} (z^{2l+1} - z_*^{2l+1}). \tag{71}$$

The boundary condition (53) leads to

$$b^{(l)} = \begin{cases} -\displaystyle\sum_{k=0}^{5m-l+1} a_k^{(l)}, & 0 \le l \le m, \\ 0, & l > m \end{cases} \tag{72}$$

for $n = 2m$ and to

$$b^{(l)} = \begin{cases} -\sum_{k=0}^{5m-l+3} a_k^{(l)}, & 0 \le l \le m, \\ 0, & l > m \end{cases} \tag{73}$$

for $n = 2m + 1$; it allows one to write $\omega_n(z, z_*)$ directly in terms of the coefficients $a_k^{(l)}$ of $\omega_n^{(1)}(z, z_*)$:

$$\omega_{2m}(z, z_*) = \sum_{l=0}^{m} (z^{2l} + z_*^{2l}) \sum_{k=0}^{5m-l+1} a_k^{(l)} ((zz_*)^k - 1), \tag{74}$$

$$\omega_{2m+1}(z, z_*) = \mathrm{i} \sum_{l=0}^{m} (z^{2l+1} - z_*^{2l+1}) \sum_{k=0}^{5m-l+3} a_k^{(l)} ((zz_*)^k - 1), \tag{75}$$

A similar situation, although somewhat more complicated, happens with $\phi_n^{(1)}(z, z_*)$. The form of $\phi_n^{(1)}(z, z_*)$ turns out to be

$$\phi_{2m}^{(1)}(z, z_*) = \mathrm{i} \sum_{l=0}^{m} (z^{2l} - z_*^{2l}) \sum_{k=0}^{5m-l} c_k^{(l)}(zz_*)^k, \tag{76}$$

$$\phi_{2m+1}^{(1)}(z, z_*) = \sum_{l=0}^{m} (z^{2l+1} + z_*^{2l+1}) \sum_{k=0}^{5m-l+2} c_k^{(l)}(zz_*)^k, \tag{77}$$

again with real coefficients $c_k^{(l)}$. The functions $\phi_n^{(0)}(z, z_*)$ have, because of (58), the same form, only the range of k is restricted to $k = 0, 1$, i.e.

$$\phi_{2m}^{(0)}(z, z_*) = \mathrm{i} \sum_{l} (z^{2l} - z_*^{2l}) (d_0^{(l)} + d_1^{(l)} zz_*), \tag{78}$$

$$\phi_{2m+1}^{(0)}(z, z_*) = \sum_{l} (z^{2l+1} + z_*^{2l+1}) (d_0^{(l)} + d_1^{(l)} zz_*). \tag{79}$$

The two sets of coefficients $d_0^{(l)}, d_1^{(l)}$ are determined by the boundary conditions (56), (57) in terms of $c_k^{(l)}$:

$$d_0^{(l)} = \begin{cases} \sum_{k} (k - 1)c_k^{(l)}, & 0 \le l \le m, \\ 0, & l > m \end{cases} \tag{80}$$

and

$$d_1^{(l)} = \begin{cases} -\sum_{k} kc_k^{(l)}, & 0 \le l \le m, \\ 0, & l > m. \end{cases} \tag{81}$$

Therefore

$$d_0^{(l)} + zz_* d_1^{(l)} = -\sum_{k} c_k^{(l)} - (zz_* - 1) \sum_{k} kc_k^{(l)} \tag{82}$$

and consequently

$$\phi_{2m}(z, z_*) = \mathrm{i} \sum_{l=0}^{m} (z^{2l} - z_*^{2l}) \sum_{k=0}^{5m-1} c_k^{(l)} ([(zz_*)^k - 1] - k(zz_* - 1)), \tag{83}$$

$$\phi_{2m+1}(z, z_*) = \sum_{l=0}^{m} (z^{2l+1} + z_*^{2l+1}) \sum_{k=0}^{5m-l+2} c_k^{(l)} ([(zz_*)^k - 1] - k(zz_* - 1)). \tag{84}$$

It might be, from the formal point of view, convenient to change the definitions of the complex variables from z, z_* to

$$Z = x_2 + \mathrm{i}x_1 (= \mathrm{i}z_*), \qquad z_* = -\mathrm{i}Z, \tag{85}$$

$$Z_* = x_2 - \mathrm{i}x_1 (= -\mathrm{i}z), \qquad z = \mathrm{i}Z_*. \tag{86}$$

In the polar co-ordinates r, ϕ related to Z, Z_* by

$$Z = re^{\mathrm{i}\varphi}, \qquad Z_* = re^{-\mathrm{i}\varphi}, \tag{87}$$

instead of the co-ordinates of Barua, r, θ $(\theta = \pi/2 - \varphi)$, which are related to z, z_* by

$$z = re^{\mathrm{i}\theta}, \qquad z_* = re^{-\mathrm{i}\theta}, \tag{88}$$

the symmetry (27), (28) expresses the fact that the streamfunction ϕ is an odd (sine) Fourier series in φ whereas ω is an even (cosine) series. For numerical work[2,3] this was of more relevance; here, however, it is not so important, because within one given order ω_n, ϕ_n the type of series is still of the same simplicity—the changes take place from one order n to the next.

This presentation is framed for application to the analytic calculation, by computer, of higher orders of the perturbation expansion. The calculations reported here have been performed partly with REDUCE and partly with MAPLE; the results are given in the following section.

5. RESULTS AND COMMENTS

The input of the computations is given by

$$\phi_0 = 0, \tag{89}$$

$$\omega_0 = c(1 - zz_*) = c(1 - r^2), \tag{90}$$

the functions of the Hagen–Poiseuille flow. The actual procedure for the computation of the terms of higher order in the expansions of ϕ and ω is displayed essentially by equations (65), (66) and (76), (77), which represent the parts $\omega_n^{(1)}(z, z_*)$, $\phi_n^{(1)}(z, z_*)$ of $\omega_n(z, z_*)$, $\phi_n(z, z_*)$ which are obtained by quadratures. These functions can be computed even separately for various pieces of the RHS of (37), (38) and then added in the end. All information on the flow is stored in the coefficients $a_k^{(l)}$ and $c_k^{(l)}$. Since the integration procedures of REDUCE and MAPLE omit integration constants, the lowest power of zz_* is $k = 1$ in $\omega_n^{(1)}(z, z_*)$ and $k = 2$ in $\phi_n^{(1)}(z, z_*)$; as a consequence,

$$a_0^{(l)} = 0, \tag{91}$$

$$c_0^{(l)} = c_1^{(l)} = 0. \tag{92}$$

Also, because of (63) one has

$$c_k^{(0)} = 0 \tag{93}$$

for even order $n = 2m$. From (74), (75) and (83), (84) it can also be seen that the functions $\omega_n(z, z_*)$ have a factor $1 - zz_*$, whereas the $\phi_n(z, z_*)$ have a factor $(1 - zz_*)^2$; these factors reflect their behaviour near the boundary.

For the orders $n \leq 6$ the coefficients are collected in Tables I and II. These tables display the fact, which follows from equations (37), (38), that the coefficients $a_k^{(l)}$ of $\omega_n(z, z_*)$ are polynomials in c of degree at most $n + 1$, with the parity of c^{n+1}, whereas the coefficients $c_k^{(l)}$ of $\phi_n(z, z_*)$ are polynomials in c, of degree n at most, with the parity of c^n. There are evidently no terms of degree zero in these polynomials. It is of course possible to present the results in terms of the numerical coefficients of these polynomials instead of by the polynomials themselves as in Tables I and II.

Table I. Coefficients $c_k^{(l)} = c(l)[k]$ of ϕ_n

```
n = 1;

c(0)[2] =  1/192*c;


n = 2;

c(1)[2] = -1/442368*c**2;
c(1)[3] =  0;
c(1)[4] =  1/8847360*c**2;


n = 3;

c(0)[2] = -17/884736*c
          -6077/356725555200*c**3;
c(0)[3] =  31/2548039680*c**3
          +1/147456*c;
c(0)[4] = -1/983040*c
          -607/101921587200*c**3;
c(0)[5] =  11/6370099200*c**3;
c(0)[6] = -1/4246732800*c**3;
c(0)[7] =  11/1664719257600*c**3;
c(1)[2] =  1/737280*c
          +19/33973862400*c**3;
c(1)[3] = -1/4423680*c
          -7/10192158720*c**3;
c(1)[4] =  1/5096079360*c**3;
c(1)[5] =  1/79272345600*c**3;
c(1)[6] = -29/4280706662400*c**3;


n = 4;

c(1)[2] =  2987729/153420526780416000*c**4
          +13213/1426902220800*c**2;
c(1)[3] = -24739/1643791358361600*c**4
          +1/8493465600*c**2;
c(1)[4] = -149/101921587200*c**2
          +77153/8218956791808000*c**4;
c(1)[5] = -35779/8218956791808000*c**4
          +1/1981808640*c**2;
c(1)[6] =  48329/32875827167232000*c**4
          -1/16986931200*c**2;
c(1)[7] = -701/1917756584755200*c**4;
c(1)[8] =  97/1598130487296000*c**4;
c(1)[9] = -17807/3797158037815296000*c**4;
c(2)[2] =  49/33973862400*c**2
          -11269/32875827167232000*c**4;
c(2)[3] =  761/7191587192832000*c**4
          -47/44590694400*c**2;
c(2)[4] =  11/31708938240*c**2
          +209/1643791358361600*c**4;
c(2)[5] = -5113/49313740750848000*c**4
          -11/285380444160*c**2;
c(2)[6] =  1/44030125670400*c**4;
c(2)[7] = -13/31642983648460800*c**4;
c(2)[8] = -19/96435759690547200*c**4;

n = 5;

c(0)[2] =  2875009/28766348771328000*c**3
          +5278042381/87486521191264419840000*c**5
          +677/20384317440*c;
c(0)[3] = -454123/6575165433446400*c**3
          -1087138037/17497304238252883968000*c**5
          -17/1132462080*c;
c(0)[4] =  90917/2739652263936000*c**3
          +62102173/1190292805323325440000*c**5
          +91/22649241600*c;
c(0)[5] = -934614823/26511067027655884800000*c**5
          -771877/82189567918080000*c**3
          -1/1698693120*c;
c(0)[6] =  1/23781703680*c
          +51830759/26511067027655884800000*c**5
          +3491/4109478395904000*c**3;
```

Table I. (Continued)

```
c(0)[7] =  593/1917756584755200*c**3
          -82534003/92788734596795596800000*c**5;
c(0)[8] =  20512459/63626560866374123520000*c**5
          -37/359579359641600*c**3;
c(0)[9] =  169/17046725197824000*c**3
          -1517407/170428288034930688000000*c**5;
c(0)[10] =  3841919/2187163029781610496000000*c**5;
c(0)[11] = -753619/3436970475371102208000000*c**5;
c(0)[12] =  4012847/31276431325877030092800000*c**5;
c(1)[2] =  16972663/21601610170682572800000*c**5
          -21157/9132174213120000*c**3
          -2123/594542592000*c;
c(1)[3] =  11111/821895679180800*c**3
          -246782603/2916217373042147328000000*c**5
          +121/101921587200*c;
c(1)[4] =  1683665281/20413521611295031296000000*c**5
          +303041/23013079017062400*c**3-1/4756340736*c;
c(1)[5] =  1/63417876480*c
          -18383/10958609055744000*c**3
          -21024973/3534808937020784640000*c**5;
c(1)[6] =  3075869/994165013537095680000*c**5
          +35671/493137407508480000*c**3;
c(1)[7] = -268411/227237717379907584000*c**5
          -4351/28766348771328000*c**3;
c(1)[8] =  7033/562541931528192000*c**3
          +3727079/11664869492168589312000*c**5;
c(1)[9] = -2998579/524919127147586519040000*c**5;
c(1)[10] =  874039/1421655969358046822400000*c**5;
c(1)[11] = -129929/4054352208909985382400000*c**5;
c(2)[2] =  6984013/874865211912644198400000*c**5
          -15787/575326975426560000*c**3
          +1/13212057600*c;
c(2)[3] =  15929/306841053560832000*c**3
          -21575921/1224811296677701877760000*c**5
          -1/47563407360*c;
c(2)[4] = -3127/65751654334464000*c**3
          +2239/144277915796766720000*c**5
          +1/570760888320*c;
c(2)[5] =  36643/1643791358361600000*c**3
          -2043919/3976660054148382720000000*c**5;
c(2)[6] = -47/84162117548113920000*c**5
          -1123/226021311774720000*c**3;
c(2)[7] =  11/2739652263936000*c**3
          +58147/795332010829676544000000*c**5;
c(2)[8] = -419/3159235487462326272000*c**5;
c(2)[9] =  839/40827043222590062592000*c**5;
c(2)[10] =  1831/2551690201411878912000000*c**5;


n = 6;

c(1)[2] = -2579669/10958609055744000*c**2
          -54501198151/388828983072286310400000*c**4
          -96737755005013/1008852568847489482673356800000*c**6;
c(1)[3] =  31468125433/29162173730421473280000*c**4
          +42833278281349/403541027538995793069342720000*c**6
          -6299/684913065984000*c**2;
c(1)[4] = -4141280096641/40761719953433918491852800000*c**6
          -5761855013/79533201082967654400000*c**4
          +262397/46965467381760000*c**2;
c(1)[5] =  230198402261/282196522754542512635904000*c**6
          +11800307/29456741141839872000*c**4
          -379/142056043315200*c**2;
c(1)[6] = -12768373/706961787404156928000*c**4
          -2072768179481/376262030339390016847872000000*c**6
          +13693/219172181114880000*c**2;
c(1)[7] = -937/1150653950853120*c**2
          +1151162782619/370382936115337047834624000000*c**6
          +370544317/556732407580773580800000*c**4;
c(1)[8] = -558791/2945674114183987200000*c**4
          -147164855797/1026169173652881864130560000*c**6
          +1649/306841053560832000*c**2;
c(1)[9] =  4153243/1093581514890805248000*c**4
          +18657306517/3527456534431781407948800000*c**6;
c(1)[10] = -206131/4374326059563220992000*c**4
          -18372099833/120941366894803933986816000000*c**6;
```

Table I. (Continued)

```
c(1)[11] =  7495489/278012722896684711936000O*c**4
           +6739098343/2058882793566305066680320000O*c**6;
c(1)[12] = -193720871/388020218787495954874368000O*c**6;
c(1)[13] =  88270541/182153936041907823260467200000*c**6;
c(1)[14] = -111459387983/4896297800806482289241358336000000*c**6;
c(2)[2] = -87161395479107/484249233046794951683211264000000*c**6
          +126392262617/15922546856810124410880000O*c**4
          -3133409/57532697542656000O*c**2;
c(2)[3] = -993199567/30620282416942546944000O*c**4
          +86048879897/194544572505025520074752000O*c**6
          +656147/14383174385664000O*c**2;
c(2)[4] =  263218913/108872115260240166912000O*c**4
          -38293/191775658475520O0*c**2
          -463018386223/7337109591618105328533504000O*c**6;
c(2)[5] = -62251/81156327635681280000*c**4
          +7703/164379135836160O0*c**2
          +941549739773/1693179136527255075815424000O*c**6;
c(2)[6] = -1033/164379135836160O0*c**2
          -160191112349/470327537924237521059840000O0*c**6
          +15208027/176740446851039232000O0*c**4;
c(2)[7] =  97/23439247147008000*c**2
          -6685771/14581086865210736640000*c**4
          +5705219/3664889905903149514752000*c**6;
c(2)[8] = -30205678093/5643930455090850252718080000O*c**6
          +10317353/7776579661445726208000O*c**4;
c(2)[9] = -1530853/758216516990958305280000*c**4
          +43899959/32086485094539819220992000O*c**6;
c(2)[10] =  13017079/106150312378734162739200000*c**4
           -4758060007/18342773979045263321333760000O0*c**6;
c(2)[11] =  905158811/252213142211872370668339200000*c**6;
c(2)[12] = -760050311/230594872879426167468195840000O*c**6;
c(2)[13] =  1189275587/7927339296543828468295532544000O*c**6;
c(3)[2] = -5761292857/171199223804422457665781760000*c**6
          +44129/106044268110623539200O*c**4
          +14213/613682107121664O0*c**2;
c(3)[3] = -12017/690392370511872O0*c**2
          +3675388723/110056643874271579928002560000*c**6
          -4896931/2449622593355403755520O0*c**4;
c(3)[4] = -487603/7953320108296765440000*c**4
          +418123451/112878609101817005054361600000*c**6
          +4457/657516543344640O0*c**2;
c(3)[5] = -373/30136174903296000*c**2
          -12820470107/3880202187874959548743680000O0*c**6
          +15964691/1458108686521073664000O0*c**4;
c(3)[6] =  19/219172181114880O0*c**2
          +206364931/806275779298692893245440000O*c**6
          -971527/124980744558949171200O0*c**4;
c(3)[7] =  1044919/379108258495479152640O0*c**4
          -24611/340311205548149597798400O*c**6;
c(3)[8] = -331687/2445703197206035109511168000*c**6
          -833033/176917187297890271232000*c**4;
c(3)[9] =  97441/324949935853267845120000*c**4
          +126630503/275141609685678949820006400000*c**6;
c(3)[10] = -6648899/917138698952263166066688000O0*c**6;
c(3)[11] =  158987/125644642274046309197414400000*c**6;
c(3)[12] =  6313493/253299598824477359341987430400O0*c**6;
```

Table II. Coefficients $a_k^{(l)} = a(l)[k]$ of ω_n

```
n = 0;

a(0)[1] = -1/2*c;


n = 1;

a(0)[1] =  1/768*c**2;
a(0)[2] = -1/1152*c**2;
a(0)[3] =  1/4608*c**2;


n = 2;

a(0)[1] =  1/589824*c**3
          +1/768*c;
a(0)[2] = -1/294912*c**3
          -1/768*c;
a(0)[3] =  1/2304*c
          +19/5308416*c**3;
a(0)[4] = -5/2359296*c**3;
a(0)[5] =  1/1474560*c**3;
a(0)[6] = -1/10616832*c**3;
a(1)[1] = -1/1152*c
          +17/26542080*c**3;
a(1)[2] = -23/35389440*c**3
          +1/3072*c;
a(1)[3] =  1/2949120*c**3;
a(1)[4] = -1/10616832*c**3;
a(1)[5] =  1/77414400*c**3;


n = 3;

a(0)[1] = -13759/1997663109120*c**4
          -41/8847360*c**2;
a(0)[2] =  12557/951268147200*c**4
          +41/10616832*c**2;
a(0)[3] = -1/663552*c**2
          -8581/535088332800*c**4;
a(0)[4] =  1/4423680*c**2
          +109/8493465600*c**4;
a(0)[5] = -1/88473600*c**2
          -2597/382205952000*c**4;
a(0)[6] =  59/25480396800*c**4;
a(0)[7] = -4679/9988315545600*c**4;
a(0)[8] =  10553/239719573094400*c**4;
a(1)[1] = -233/1630745395200*c**4
          -1/3932160*c**2;
a(1)[2] =  1553/7134511104000*c**4
          +19/44236800*c**2;
a(1)[3] = -19/101921587200*c**4
          -7/26542080*c**2;
a(1)[4] =  1/10569646080*c**4
           +13/247726080*c**2;
a(1)[5] = -157/5707608883200*c**4;
a(1)[6] =  1/244611809280*c**4;
a(1)[7] = -11/39953262182400*c**4;


n = 4;

a(0)[1] = -23497/2497078886400*c**3
          -77/35389440*c
          -318853/5114017559347200 0*c**5;
a(0)[2] =  54491/2853804441600*c**3
          +8165509/460261580341248000*c**5
          +49/17694720*c;
a(0)[3] = -17/10616832*c
          -43259179/13807847410237440 00*c**5
          -359/16721510400*c**3;
a(0)[4] =  169387/4383443622297600*c**5
          +2407/163074539520*c**3
          +1/2359296*c;
```

Table II. (Continued)

```
a(0)[5] = -1/19660800*c
          -27/4194304000*c**3
          -287417/8218956791808000*c**5;
a(0)[6] =  926227/39450992600678400*c**5
          +4313/2446118092800*c**3;
a(0)[7] = -101/356725555200*c**3
          -41633/3595793596416000*c**5;
a(0)[8] =  7529357/18410463213649920000*c**5
          +1187/53271016243200*c**3;
a(0)[9] = -272143/27615694820474880O*c**5;
a(0)[10] =  67519/4602615803412480000*c**5;
a(0)[11] = -7489/7232681976791040000*c**5;
a(1)[1] = -301085189/455658964537835520000*c**5
          +11/5308416*c
          -43553/1712282664960O*c**3;
a(1)[2] = -103/70778880*c
          +47413907/33752515891691520000*c**5
          +20987/5707608883200*c**3;
a(1)[3] =  1/2211840*c
          -16910797/86299046313984000O*c**5
          -32357/1070176665600O*c**3;
a(1)[4] =  717281/375723739054080000*c**5
          -1/17694720*c
          +797/489223618560*c**3;
a(1)[5] = -1899391/14383174385664000O*c**5
          -29/47563407360*c**3;
a(1)[6] =  1022501/15780397040271360O*c**5
          +407/2853804441600*c**3;
a(1)[7] = -9203851/41423542230712320000*c**5
          -1973/119859786547200*c**3;
a(1)[8] =  470909/920523160682496000*c**5;
a(1)[9] = -411557/5695737056722944000*c**5;
a(1)[10] =  2227487/455658964537835520000*c**5;
a(2)[1] = -1/9830400*c
          +41/3567255552000*c**3
          -400847/949289509453824000O*c**5;
a(2)[2] =  7408013/91131792907567104000*c**5
          +1/17694720*c
          -127/856141332480*c**3;
a(2)[3] =  1103/4280706662400*c**3
          -569/6903923705118720*c**5
          -1/123863040*c;
a(2)[4] =  86503/18410463213649920000*c**5
          -4273/22830435532800*c**3;
a(2)[5] = -4567/295882444505088000*c**5
          +173/2853804441600*c**3;
a(2)[6] = -1219/171228266496000*c**3
          +19813/591764889010176000O*c**5;
a(2)[7] = -1073/1518863215126118400*c**5;
a(2)[8] =  1523/12150905721008947200*c**5;
a(2)[9] = -13/142393426418073600O*c**5;


n = 5;

a(0)[1] =  15087160573/30377264302522368000O*c**4
          +12919/114152177664O*c**2
          +667323147359359/2101776185098936422236160000O*c**6;
a(0)[2] = -11533/114152177664O*c**2
          -24579938672753/3002537407284194888908800000*c**6
          -23084993/241089399226368000*c**4;
a(0)[3] =  673775311/55231389640949760000*c**4
          +74113/1712282664960O*c**2
          +50521846099/34994608476505767936000O*c**6;
a(0)[4] = -287/407686348800*c**2
          -131031733387/69989216953011535872000O*c**6
          -28355737/2630066173378560000*c**4;
a(0)[5] = -1/27179089920*c**2
          +649999060793/34994608476505767936000000*c**6
          +66449911/98627481501696000O*c**4;
a(0)[6] = -319408168483/22269296303230943232000000*c**6
          -12898177/4314952315699200000*c**4
          +1/28538044416*c**2;
a(0)[7] = -29/665887703040O*c**2
          +25633669567/29692395070974590976000O*c**6
          +8596747/920523160682496000*c**4;
```

Table II. (Continued)

```
a(0)[8] = -107238710987/2672315556387713187840000*c**6
          -21941419/110462779281899520000*c**4;
a(0)[9] =  5400353723/381759365198244741120000*c**6
          +7429/28179280429056000*c**4;
a(0)[10] = -5284777/30377264302522368000*c**4
           -19261211351/5249191271475865190400000*c**6;
a(0)[11] =  3836772169/5774110398623451709440000*c**6;
a(0)[12] = -5687048243/7506343518210487222720000*c**6;
a(0)[13] =  37884152297/91076968020953911630233600000*c**6;
a(1)[1] =  1295086893481/350296030849822737039360000*c**6
          +6979807/78901985201356800000*c**4
          +59807/57076088832000*c**2;
a(1)[2] = -2633/132120576000*c**2
          -1742231923/15188632151261184000000*c**4
          -2019551572309/212300624757468325478400000*c**6;
a(1)[3] = -104927/6575165433446400000*c**4
          +5467599307/34994608476505767936000*c**6
          +9581/6115295232000*c**2;
a(1)[4] = -35064871627/1959698074684323004416000*c**6
          +308129/18410463213649920*c**4
          -479/815372697600*c**2;
a(1)[5] =  728404709377/48992451867108075110400000*c**6
          +31/271790899200*c**2
          -4189117/2301307901706240000*c**4;
a(1)[6] =  303697/2958824445050880000*c**4
          -3486897487/381759365198244741120000*c**6
          -61/5707608883200*c**2;
a(1)[7] = -2312647/6903923705118720000*c**4
          +387185963/92788734596795596800000*c**6;
a(1)[8] =  201521/3375251589169152000*c**4
          -20126441/14283513663879905280000*c**6;
a(1)[9] = -805171/18226358581513420800*c**4
          +2511673/729054343260536832000*c**6;
a(1)[10] = -12617861/21663329056884523008000*c**6;
a(1)[11] =  21583333/3502960308498227370393600*c**6;
a(1)[12] = -660544553/2101776185098936422236160000*c**6;
a(2)[1] = -2579/17122826649600*c**2
          +491589127/7642822491268859717222400*c**6
          -1841839/45565896453783552000*c**4;
a(2)[2] =  3701/19976631091200*c**2
          -149680123/1592254685681012441088000*c**6
          +9257291/141760566745104384000*c**4;
a(2)[3] = -65239/276156948204748800*c**4
          -1243/11415217766400*c**2
          +42439477/839870603436138430464000*c**6;
a(2)[4] = -1573447/331388337845698560000*c**4
          +17/570760888320*c**2
          +25361387/4409320668039726759936000*c**6;
a(2)[5] = -17/5707608883200*c**2
          -496183/6362656086637412352000*c**6
          +123371/197254963003392000*c**4;
a(2)[6] = -99619/954398412995611852800*c**6
          -134809/4339609186074624000*c**4;
a(2)[7] =  284983/26245956357379325952000*c**6
          +106627/15188632151261184000*c**4;
a(2)[8] = -44879/752198925586268160000*c**4
          -489287/129979974341307138048000*c**6;
a(2)[9] =  577519/955352811408607464652800*c**6;
a(2)[10] = -200047/367443389003310563328000*c**6;
a(2)[11] =  138031/431133576430551060971520000*c**6;


n = 6;

a(0)[1] =  37770754488218911/1468889340241944686772407500800000*c**7
          +125657331751657 9/2101776185098936422236160000000*c**5
          +38142227/1150653950853120000*c**3
          +18689/5707608883200*c;
a(0)[2] = -126680105105857189/1468889340241944686772407500800000*c**7
          -33081041929399/20016916048561299259392000*c**5
          -27230849/409121404747776000*c**3
          -12403/2853804441600*c;
a(0)[3] =  358884211898426713/188857200888250031156452392960000*c**7
          +7524455441/25921932204819087360000*c**5
          +17459459/2301307901706240000*c**3
          +677/244611809280*c;
```

Table II. (Continued)

```
a(0)[4] = -30058901688539597/96849846609358990336642252800000*c**7
          -514023274183/1399784339060230717440000*c**5
          -1071989/19481971654656000*c**3
          -17/18119393280*c;
a(0)[5] =  381061060458809/960811970330942364450816000000*c**7
          +137138251501/388828983072286310400000*c**5
          +2936189/109586090557440000*c**3
          +91/452984832000*c;
a(0)[6] = -23403537247/5759112709276377808896000O*c**7
          -27673898003/106044268110623539200000*c**5
          -17476757/19725496300339200000*c**3
          -1/40768634880*c;
a(0)[7] =  19306857721/571578605116260876288000000*c**7
          +441163397/29692395070974590976000*c**5
          +44813/23013079017062400*c**3
          +1/665887703040*c;
a(0)[8] = -60225631441663/2633834212375730117935104000000*c**7
          -38371779521/593847901419491819520000*c**5
          -81839/30684105356083200O*c**3;
a(0)[9] =  1270052881681/10101352803145555850035200000*c**7
          +24247357847/114527809559473422336000O*c**5
          +3821/207117711153561600*c**3;
a(0)[10] = -5713428901249/10261691736528818641305600000O*c**7
           -9731674969/190879682599122370560000O*c**5
           -869/204560702373888000O*c**3;
a(0)[11] =  150011512363/766459691432090775060480000O*c**7
           +1324745197/1539762772966253789184000*c**5;
a(0)[12] = -34121982233293/638570417204564771450388480000O*c**7
           -212968333/230964415944938068377600O*c**5;
a(0)[13] =  49258775523233/449660002114881026562981888000O0*c**7
           +3166210829/6505497715782422259302400O0*c**5;
a(0)[14] = -9365657642351/5875557360967778747089630003200O*c**7;
a(0)[15] =  13571117865211/9180558376512154292327546880000O0*c**7;
a(0)[16] = -650456415029/979259560161296457848271667200000*c**7;
a(1)[1] = -142909/42807066624000*c
          +894839317/1518863215126118400O0*c**3
          +24200242656896 71/612037225100810286155169792000O0*c**7
          +14662211698776 1/2627220231373670527795200000*c**5;
a(1)[2] =  298909/114152177664000*c
          -493511/390654119116800O0*c**3
          -27202019766170023/251809601184333374875269857280000*c**7
          -7902926006 5163/636901874272404976435200000*c**5;
a(1)[3] =  138223147/986274815016960000O*c**3
          +192263103553/107126352479099289600000O*c**5
          +16373073887687497/807082055077991586138685440000O0*c**7
          -3739/3567255552000*c;
a(1)[4] = -27474852797855971/96849846609358990336642252800000*c**7
          -1913982182653/1049838254295173038080000O*c**5
          -39803501/394509926006784000O*c**3
          +197/815372697600*c;
a(1)[5] =  11870809/230130790170624000O*c**3
          +102187541227/765507060423563673600000*c**5
          +15802135228986779/513597671413267372997345280000O0*c**7
          -1/31708938240*c;
a(1)[6] = -18049591141/254506243465496494080000*c**5
          -1690841/920523160682496000*c**3
          +1/507343011840*c
          -71745122888687/2709086618443608121304678400O0*c**7;
a(1)[7] =  18192585203/668078889096928296960000*c**5
          +324996330165719/177783809335361782960619520000O*c**7
          +24151/5523138964094976O*c**3;
a(1)[8] = -23357/368209264272998400*c**3
          -16515374047/2226929630323094323200000*c**5
          -2400348069638989/2370450791138157106141593600000O*c**7;
a(1)[9] =  2893885451/209967650859034607616000O*c**5
          +15147141073241/338635827305451015163084800000*c**7
          +443/9375698858803200O*c**3;
a(1)[10] = -1756554327827/1128786091018170050543616000000*c**7
           -684922547/4199353017180692152320000*c**5;
a(1)[11] =  9637894088467/230594872879426167468195840000O*c**7
           +22940179/2085095421725135339520000*c**5;
a(1)[12] = -2709783485581/3228328220311966344554741760000*c**7
           -162846577/4003383209712259851878400O0*c**5;
a(1)[13] =  9388848418427/786905003701041796485218304000O0*c**7;
a(1)[14] = -3170521405273/29377786804838893735448150016000O*c**7;
a(1)[15] =  9848580420563/20809265653427549729275772928000000*c**7;
```

Table II. (Continued)

```
a(2)[1] =  81778377900855289/49942237568226119350261855027200000*c**7
           -574601707/31592354874623262720000000*c**5
           +10789/47563407360000*c
           -8276561/361634098839552000000*c**3;
a(2)[2] =  2769522791323/191070562281721492930560000*c**5
           +122430367/86792183721492480000*c**3
           -2293/14269022208000*c
           -41806736418886 7689/864384880988528988754532106240000000*c**7;
a(2)[3] =  29138255610767/3254154846074462075311179694080*c**7
           -7334449/3020466620989440000*c**3
           -108314839499/30960507777130797465600000*c**5
           +139/2853804441600*c;
a(2)[4] =  75519049/36820926427299840000*c**3
           -952898546313047/821756274261227796795752448000000*c**7
           +80621886389/17815437042584754585600000*c**5
           -1/126835752960*c;
a(2)[5] =  6103004972873/5502832193713578996400128000000*c**7
           -30155546363/816540864451801251840000*c**5
           -5651/5870683422720000*c**3+1/1902536294400*c;
a(2)[6] = -40905862011781/50795374095817652274462720000000*c**7
           +486446609/238599603248902963200000*c**5
           +29659/112717121716224000*c**3;
a(2)[7] = -9492439541/122481129667770187776000000*c**5
           +5235637371919/116406065636248786462310400000*c**7
           -26167/632859672969216000*c**3;
a(2)[8] =  491/162012076280119296*c**3
           -5258088970489/270908661844360812130467840000*c**7
           +26677361/133312794196212449280000*c**5;
a(2)[9] =  42609087071/660339863245629479568015360000*c**7
           -26389637/7798798460478428282880000*c**5;
a(2)[10] =  7389133/21230062475746832547840000*c**5
            -25665316613/15722377696324511418286080000*c**7;
a(2)[11] =  337085538733/110056643874271579928002560000000*c**7
            -21466513/125105725303508120371200000*c**5;
a(2)[12] = -1573189189931/387399386437435961346569011200000*c**7;
a(2)[13] =  136373189873/39636696482719142341477662720000*c**7;
a(2)[14] = -67657016129/47564035779262970809773195264000000*c**7;
a(3)[1] =  3154639/47253522248368128000*c**3
           -296142128401597/403379611127980194752114982912000000*c**7
           +100361837/24496225933554037555200000*c**5
           -17/4994157772800*c;
a(3)[2] =  153258125266189/92201053972109758800483424665600000*c**7
           -409228973/2038085997671695924592640 0*c**5
           -418343/300022363481702400 0*c**3
           +1/422785843200*c;
a(3)[3] =  2564083/16569416892284928000*c**3
           -127791991/10699951487776403604111360 0000*c**7
           +64962211/17637282672158907039744 00*c**5
           -1/1712282664960*c;
a(3)[4] = -644799193/400205977724623927010918400000*c**7
           -449009/4602615803412480000*c**3
           -713772959/24496225933554037555200000*c**5
           +1/2283043553280 0*c:
a(3)[5] =  250247/72326819767910400 00*c**3
           +3466147/17497304238252883968000 00*c**5
           +38688630727/93124852508999029169848320000 00*c**7;
a(3)[6] =  53369233/381759365198244741120 0000*c**5
           -122258375731/31928520860228238572519424000000*c**7
           -27719/433960918607462400 0*c**3;
a(3)[7] =  5867/119668010888724480 00*c**3
           -16596103/151643303398191661056 0000*c**5
           +21303467/114642337369032895758336 00000*c**7;
a(3)[8] =  106661749/28306749967662443397120000*c**5
           -3387067433/684796895217689830663127040 0000*c**7;
a(3)[9] = -1067747/17059871632296561868800 00*c**5
           +48383059/660339863245629479568015360000*c**7;
a(3)[10] = -169730633/1956562557764828087608934400 0000*c**7
            +34644671/849202499029873301913600000*c**5;
a(3)[11] =  249410323/182938599151011426191435366400000*c**7;
a(3)[12] = -31383631/253299598824477359341987430400000*c**7;
a(3)[13] =  125659/473977279618529604223264358400 00*c**7;
```

Table III. Functions ϕ_1, ϕ_2 in polar co-ordinates r, θ related to z, z_*

$$\phi_0 = 0$$

$$\phi_1 = (1 - r^2)^2 \frac{c}{2^5 \times 3} r \cos\theta,$$

$$\phi_2 = (1 - r^2)^2 \frac{c^2}{2^{15} \times 3^3 \times 5}(-r^4 - 2r^2 + 17) r^2 \sin 2\theta$$

Table IV. Functions ω_0, ω_1, ω_2 in polar co-ordinates r, θ related to z, z_*

$$\omega_0 = (1 - r^2)c$$

$$\omega_1 = (1 - r^2)\frac{c^2}{2^8 \times 3^2}(r^4 - 3r^2 + 3) r \sin\theta$$

$$\omega_2 = (1 - r^2)\left[\left(-\frac{c}{2^7 \times 3^2}(1 - r^2)^2 + \frac{c^3}{2^{17} \times 3^4 \times 5}(1 - r^2)^3(-10r^4 + 32r^2 - 37)\right)\right.$$
$$\left. + \left(\frac{c}{2^9 \times 3^2}(-3r^2 + 5) + \frac{c^3}{2^{17} \times 3^4 \times 5^2 \times 7}(-48r^8 + 302r^6 - 958r^4 + 1457r^2 - 923)\right) r^2 \cos 2\theta\right]$$

There are very large numbers involved in the denominators. They come from the iterated powers 2^2 and 2^4 in the operators Δ and Δ^2 of (20) and (21) and from the integration of powers of z and z_*.

The functions ω_1, ω_2 and ϕ_1, ϕ_2 are given in Tables III and IV for comparison with Barua.[4] One has thereby to notice that his constant c is in the present notation is equal to $-4c$. The second-order perturbation of the axial flow, ω_2, has been calculated only partly by Barua; in the notation

$$\omega_2(r, \theta) = f_1(r) + f_2(r)\cos 2\theta \tag{94}$$

he computed the term $f_1(r)$, which contributes to the total flow rate. The computation of the other term is tedious if performed by hand but is straightforward by computer.

A detailed physical analysis of the flow will be given in a separate publication. It should be stressed that such an analysis would not be possible without the use of computer algebra. The present paper illustrates the power of computer algebra as a tool of analytic fluid mechanics.

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